ON ORTHOGONAL 3D PROLATE SPHEROIDAL MONOGENICS

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Abstract. Complete orthogonal systems of monogenic polynomials over 3D prolate spheroids have recently experienced an upsurge of interest due to their many remarkable properties. These generalized polynomials and their applications to the theory of quasiconformal mappings and approximation theory have played a major role in this development. In particular, the underlying functions are of three real variables and take on values in the reduced quaternions (identified, respectively, with $\mathbb{R}^3$), and are generally assumed to be null-solutions of the well-known Riesz system in $\mathbb{R}^3$. This paper introduces and explores a new complete orthogonal system of monogenic functions as solutions to this system for the space exterior of a 3D prolate spheroid. This will be done in the linear spaces of square integrable functions over $\mathbb{R}$. The representations of these functions are explicitly given. Some important properties of the system are briefly discussed, from which several recurrence formulae for fast computer implementations can be derived.

1. Introduction

A prolate spheroid is a quadric surface generated by rotating an ellipse about its major axis. In general, the boundary value problems involving prolate spheroidal bodies (including their limiting configurations – the sphere, and thin circular disk) are treated in prolate spheroidal coordinates (μ, θ, φ). Here μ is the radial term with $\mu > 0$, $\theta \in [0, \pi)$ is the asymptotic angle with respect to the major axis, and $\varphi \in [0, 2\pi)$ is the rotation term.

In prolate spheroidal coordinates (cf. E. Hobson [13], N. Lebedev [16]), the Cartesian coordinates may be parameterized by $x = x(\mu, \theta, \varphi)$ so that

$$x_0 = ca\cos\theta, \qquad x_1 = cb\sin\theta\cos\varphi, \quad x_2 = cb\sin\theta\sin\varphi \tag{1.1}$$

where $c > 0$ is the eccentricity of the generating ellipse, and $a = \cosh\mu$, $b = \sinh\mu$, are respectively, the semimajor and semiminor axis of this ellipse. For simplicity we choose $c = 1$. Using these transformation relations the surfaces of revolution for which μ is the parameter consist of the confocal prolate spheroids

$$\mathcal{S} : \frac{x_0^2}{a^2} + \frac{x_1^2 + x_2^2}{b^2} = 1. \tag{1.2}$$

Equation (1.2) represents a prolate spheroid in Cartesian coordinates. Accordingly, the surface of $\mathcal{S}$ is matched with the surface of the supporting spheroid $\mu = \mu_0$ if we put $\cosh^2\mu_0 = a^2$ and $\sinh^2\mu_0 = b^2$.

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The Laplace equation in prolate spheroidal coordinates (1.1) is given by

$$\Delta_3 U = \frac{1}{\sin^2\theta + \sinh^2\mu}\left(\frac{\partial^2 U}{\partial\mu^2} + \frac{\partial^2 U}{\partial\theta^2} + \coth\mu\frac{\partial U}{\partial\mu} + \cot\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{\sin^2\theta\sinh^2\mu}\frac{\partial^2 U}{\partial\varphi^2} = 0.$$

The previous equation is separable in prolate spheroidal coordinates. The corresponding solutions are the well-known *prolate spheroidal harmonics* [7, 13], which are a combination of products of spherical functions: $U := \Xi(\mu)\,\Theta(\theta)\,\Phi(\varphi)$, where $\Xi(\mu)$, $\Theta(\theta)$ and $\Phi(\varphi)$ satisfy the differential equations

$$\begin{aligned}
\frac{d^2\Xi(\mu)}{d\mu^2} + \coth\mu\frac{d\Xi(\mu)}{d\mu} - \left[\frac{l^2}{\sinh^2\mu} + n(n+1)\right]\Xi(\mu)\sinh\mu &= 0,\\
\frac{d^2\Theta(\theta)}{d\theta^2} + \cot\theta\frac{d\Theta(\theta)}{d\theta} + \left[n(n+1) - \frac{l^2}{\sin^2\theta}\right]\Theta(\theta)\sin\theta &= 0,\\
\frac{d^2\Phi(\varphi)}{d\varphi^2} + l^2\Phi(\varphi) &= 0,
\end{aligned} \tag{1.3}$$

where n is a constant and l is a parameter introduced during the separation of variables method. The periodicity of Φ requires that l is a positive integer or zero. Hence solutions to the equation (1.3) are either $\cos(l\varphi)$ or $\sin(l\varphi)$. The solutions $\Theta(\theta)$ and $\Xi(\mu)$ are given, respectively, by $P_n^l(\cos\theta)$ or $Q_n^l(\cos\theta)$, and $P_n^l(\cosh\mu)$ or $Q_n^l(\cosh\mu)$. Here P_n^l and Q_n^l are the *Ferrer's associated Legendre functions* of the first and second kinds of n-th degree and l-th order. In this assignment, the sign convention of including the Condon-Shortley phase is adopted (even though the reader should pay attention that in the topic of associated Legendre functions different authors may employ different conventions). When the argument is greater than unity, we define the Ferrer functions as

$$\left.\begin{matrix} P_n^l(\cosh\mu) \\ Q_n^l(\cosh\mu) \end{matrix}\right\} := (-1)^l(\sinh\mu)^l\,\frac{d^l}{dt^l}\left\{\begin{matrix} P_n(t) \\ Q_n(t) \end{matrix}\right.\Bigg|_{t=\cosh\mu}$$

because these definitions avoid imaginary values when l is odd. The general theory and background on associated Legendre functions is contained in the monograph by W.W. Bell [1] (cf. [13]).

The original impetus of the investigation of sets of orthogonal harmonic polynomials over prolate and oblate spheroids has been developed by P. Garabedian [7]. The orthogonality was taken in several different norms, each of which lead to the discussion of a partial differential equation by means of the kernel of the orthogonal system corresponding to that norm. Spheroidal functions usually appear in the solutions of Dirichlet problems in spheroidal domains arising in hydrodynamics, elasticity and electromagnetism. For the solvability of boundary value problems of radiation, scattering, and propagation of acoustic signals and electromagnetism waves radiated by sources with spheroidal shapes, spheroidal functions are frequently encountered. These applications have stimulated a surge of new ideas and methods, both theoretical and applied, and have reawakened an interest in spectral analysis, signal processing, optical system analysis, approximation theory, potential theory, the theory of partial differential equations, and so forth.

Recently, multidimensional extensions of prolate spheroidal harmonics to the quaternionic analysis setting have been introduced in [21, 23], which have provided many of their properties and have subsequently attracted special attention. In [24] it has been shown that the underlying prolate spheroidal monogenics play an important role in defining the monogenic Szegö kernel function for 3D (prolate) spheroids. These results have been used to investigate a particular class of approximation properties for monogenic functions over (prolate) spheroids in terms of special systems [26]. Extensions of the prolate spheroidal functions and their connections with the finite quaternionic and Clifford Fourier transform setting were introduced in [27] and [14]. These generalized spheroidal functions have been successfully applied to analyse the energy concentration problem introduced in the early-sixties by D. Slepian and H.O. Pollak [32]. In this line of research, in [29] the authors have exploited a complete orthogonal system of 3D oblate spheroidal monogenics by means of harmonic functions and some recurrence formulae have been found. We honestly expect that the rising popularity of the spheroidal functions is still on the rise and will likely see even more growth in the future, due to their promising applications in many fields.

Regarding the organization of the paper, in the next section we summarize some definitions and basic properties of quaternionic analysis. Section 3 recalls a complete orthogonal system of monogenic polynomials over 3D prolate spheroids. The former were introduced in the papers [21, 23] which could be explicitly expressed in terms of products of Ferrer's associated Legendre functions multiplied by Chebyshev polynomial factors (see expressions (3.1)-(3.3) below). In particular, the underlying functions are of three real variables and take values in the reduced quaternions (identified with $\mathbb{R}^3$), and are generally assumed to be null-solutions of the Riesz system in $\mathbb{R}^3$. With the help of these polynomials, in Section 4 we construct a new complete orthogonal system of monogenic functions as solutions to the Riesz system for the space exterior of 3D (prolate) spheroids. This will be done in the spaces of square integrable functions over $\mathbb{R}$. Some important properties, and efficient recurrence formulae for the basis functions are discussed. This helps to reduce the amount of calculations in practice. The final section shows the concluding remarks and discusses how the used methods can be extended within this context. Most relevant to our study are the intimate connections between monogenic functions and spheroidal structures [3, 10, 11, 19, 22, 25, 28], and the potential flexibility afforded by a spheroid's non-spherical canonical geometry. We will not consider concrete applications of the (prolate) spheroidal monogenics in this paper.

The motivation for writing the present paper is to develop further general numerical methods to solve both basic initial-boundary value and conformal mapping problems. The topic here is the prolate functions, but the principles can be extended to oblate spheroids as well (see Remark 3.1 below). Besides its obvious importance this case will not be discussed in the present article. Further investigations will be reported in a forthcoming paper.

2. Preliminaries

2.1. **The Riesz system revisited.** As is well-known, a holomorphic function $f(z) = u(x,y) + iv(x,y)$ of a complex variable $z = x + iy$ ($i = \sqrt{-1}$) defined in an open domain of the complex plane, satisfies the Cauchy-Riemann system

$$\begin{cases} \dfrac{\partial u}{\partial x} &= \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} &= -\dfrac{\partial v}{\partial x} \end{cases}.$$

As in the case of two variables, we may now characterize a possible analogue of the Cauchy-Riemann system in an open domain of the Euclidean space $\mathbb{R}^3$. More precisely, consider the pair $f = (f_0, f^*)$ where f_0 is a real-valued continuously differentiable function defined on an open domain $\Omega \subset \mathbb{R}^3$ and $f^* = (f_1, f_2, f_3)$ is a continuously differentiable vector-field in Ω for which

$$\text{(R)} \quad \begin{cases} \text{div } f^* &= 0 \\ \text{rot } f^* &= 0 \end{cases}. \tag{2.1}$$

Recall that the 3-tuple f^* is said to be an *M. Riesz system of conjugate harmonic functions* in the sense of E.M. Stein and G. Weiß [33, 34], and system (R) is called the *Riesz system* [31]. The Riesz system has a physical relevance as it describes the velocity field of a stationary flow of a non-compressible fluid without sources nor sinks.

2.2. **Quaternionic analysis.** The (R)-system can be obtained by using a quaternion framework. Throughout the paper, let

$$\mathbb{H} := \{\mathbf{z} = z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k} : \ z_l \in \mathbb{R}, \, l = 0, 1, 2, 3\}$$

be the Hamiltonian *skew field of quaternions*, where the imaginary units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ obey the laws of multiplication: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$; $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, and $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$.

In the sequel, let $\mathcal{A} := \text{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}\} \subset \mathbb{H}$ be the space of *reduced quaternion* elements of the form $\mathbf{x} := x_0 + x_1\mathbf{i} + x_2\mathbf{j}$, emphasizing that $\mathcal{A}$ is a real vectorial subspace, but not a subalgebra, of $\mathbb{H}$. The real vector space $\mathbb{R}^3$ is embedded in $\mathcal{A}$ via the identification

$$x := (x_0, x_1, x_2) \in \mathbb{R}^3 \quad \leftrightarrow \quad \mathbf{x} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} \in \mathcal{A}.$$

As a matter of fact, throughout the text we will often use the symbol x to represent a point in $\mathbb{R}^3$ and $\mathbf{x}$ to represent the corresponding reduced quaternion. The scalar and vector parts of $\mathbf{x}$, $\text{Sc}(\mathbf{x})$ and $\text{Vec}(\mathbf{x})$, are defined as the x_0 and $x_1\mathbf{i} + x_2\mathbf{j}$ terms, respectively. Like in the complex case, the conjugate of $\mathbf{x}$ is the reduced quaternion $\overline{\mathbf{x}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j}$, and the norm $|\mathbf{x}|$ of $\mathbf{x}$ is defined by $|\mathbf{x}|^2 = \mathbf{x}\overline{\mathbf{x}} = \overline{\mathbf{x}}\mathbf{x} = x_0^2 + x_1^2 + x_2^2$.

Now, let Ω be an open subset of $\mathbb{R}^3$. We say that

$$\mathbf{f} : \Omega \to \mathcal{A}, \qquad \mathbf{f}(x) := \text{Sc}(\mathbf{f}(x)) + [\mathbf{f}(x)]_1\mathbf{i} + [\mathbf{f}(x)]_2\mathbf{j} \tag{2.2}$$

is a reduced quaternion-valued function or, briefly, an $\mathcal{A}$-valued function, where the components $\text{Sc}(\mathbf{f})$ and $[\mathbf{f}]_m$ $(m = 1, 2)$ are scalar-valued functions defined in Ω. Continuity, differentiability, integrability, and so on, which are ascribed to $\mathbf{f}$ are defined componentwise.

We will work with the real linear Hilbert space of all $\mathcal{A}$-valued functions in Ω that we denote by $L^2(\Omega; \mathcal{A}; \mathbb{R})$, for which $\int_\Omega |\mathbf{f}(x)|^2 dV < \infty$, where dV denotes the Lebesgue measure on Ω normalized so that $V(\Omega) = 1$. In this assignment, the scalar inner product is defined by

$$< \mathbf{f}, \mathbf{g} >_{L^2(\Omega;\mathcal{A};\mathbb{R})} = \int_\Omega \mathrm{Sc}(\overline{\mathbf{f}}\,\mathbf{g})\, dV. \tag{2.3}$$

To simplify matters further we shall remark that using the embedding of $\mathbb{R}$ in $\mathcal{A}$ the inner product of two scalar-valued functions $f, g : \Omega \to \mathbb{R}$ can also be written by using the scalar inner product (2.3), and it will be denoted simply by $< f, g >_{L^2(\Omega)}$. The reader should note that the norm induced by the scalar inner product (2.3),

$$\|\mathbf{f}\|^2_{L^2(\Omega;\mathcal{A};\mathbb{R})} := < \mathbf{f}, \mathbf{f} >_{L^2(\Omega;\mathcal{A};\mathbb{R})} = \int_\Omega |\mathbf{f}(x)|^2 dV$$

coincides with the L^2-norm for $\mathbf{f}$, considered as a vector-valued function.

For a real-differentiable $\mathcal{A}$-valued function $\mathbf{f}$ that has continuous first partial derivatives, the (reduced) quaternionic operators

$$\overline{\partial} = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2}, \tag{2.4}$$

and

$$\partial = \frac{\partial}{\partial x_0} - \mathbf{i}\frac{\partial}{\partial x_1} - \mathbf{j}\frac{\partial}{\partial x_2}, \tag{2.5}$$

are called, respectively, *generalized* and *conjugate generalized Cauchy-Riemann operators* on $\mathbb{R}^3$. They correspond to the 3D extensions of the classical Cauchy-Riemann operator $\partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ and its conjugate $\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$.

In prolate spheroidal coordinates, $\overline{\partial}$ has the form

$$\begin{aligned}\overline{\partial} =& \frac{1}{\sin^2\theta + \sinh^2\mu}\left(\cos\theta\sinh\mu\frac{\partial}{\partial\mu} - \sin\theta\cosh\mu\frac{\partial}{\partial\theta}\right)\\ &+ \frac{1}{\sin^2\theta + \sinh^2\mu}(\cos\phi\mathbf{i} + \sin\phi\mathbf{j})\left(\sin\theta\cosh\mu\frac{\partial}{\partial\mu} + \cos\theta\sinh\mu\frac{\partial}{\partial\theta}\right)\\ &+ \frac{1}{\sin\theta\sinh\mu}(-\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j})\frac{\partial}{\partial\varphi}.\end{aligned}$$

We further assume the reader to be familiar with the fact that the operators (2.4) and (2.5) factorize the Laplace operator in $\mathbb{R}^3$ in the sense that $\Delta_3 = \overline{\partial}\partial = \partial\overline{\partial}$. It is of interest to remark at this point that the operator $\overline{\partial}$ in (prolate) spheroidal coordinates reduces to the operator $\overline{\partial}$ in spherical coordinates if $\cosh\mu \simeq \sinh\mu$, which occurs as μ appoaches infinity, and in which case the two foci coincide at the origin.

A continuously real-differentiable $\mathcal{A}$-valued function $\mathbf{f}$ is said to be two-sided monogenic (or simply *monogenic*) in Ω if

$$\overline{\partial}\mathbf{f} = 0 = \mathbf{f}\overline{\partial}$$

in Ω, which is equivalent to the *Riesz system*

$$\text{(R)}\quad \begin{cases} \dfrac{\partial \mathrm{Sc}(\mathbf{f})}{\partial x_0} - \dfrac{\partial[\mathbf{f}]_1}{\partial x_1} - \dfrac{\partial[\mathbf{f}]_2}{\partial x_2} = 0\,, \\[2ex] \dfrac{\partial \mathrm{Sc}(\mathbf{f})}{\partial x_1} + \dfrac{\partial[\mathbf{f}]_1}{\partial x_0} = 0, \quad \dfrac{\partial \mathrm{Sc}(\mathbf{f})}{\partial x_2} + \dfrac{\partial[\mathbf{f}]_2}{\partial x_0} = 0, \quad \dfrac{\partial[\mathbf{f}]_1}{\partial x_2} - \dfrac{\partial[\mathbf{f}]_2}{\partial x_1} = 0\,. \end{cases}$$

This system can also be written in abbreviated form:

$$\begin{cases} \operatorname{div} \overline{\mathbf{f}} &= 0 \\ \operatorname{rot} \overline{\mathbf{f}} &= 0. \end{cases}$$

Following [17], the solutions of the system (R) are customary called (R)-solutions. Denote by $\mathcal{M}(\Omega, \mathcal{A})$ the space of all square integrable $\mathcal{A}$–valued monogenic functions in Ω. The following notation $\Omega^- := \mathbb{R}^3 \backslash \overline{\Omega}$ is also used where $\overline{\Omega}$ means the closure of Ω. From now on, we always consider Ω as a domain bounded by a prolate spheroid, particularly $\Omega := \{(\mu, \theta, \varphi) \,|\, 0 \leq \mu < \mu_0,\, 0 \leq \theta < \pi,\, 0 \leq \varphi < 2\pi\} \subset \mathbb{R}^3$.

3. 3D inner solid prolate spheroidal monogenics revisited

In this section we recall an explicit complete system of inner solid prolate spheroidal monogenics in 3D required for subsequent derivations. Unless stated otherwise, all these facts can be found in [21, 23].

By virtue of [7] (cf. [13]), scalar-valued harmonic functions in prolate spheroidal coordinates (1.1) are given by

$$U_{n,l}(\mu, \theta) \cos(l\varphi), \qquad U_{n,l}(\mu, \theta) \sin(l\varphi)$$

where $U_{n,l}(\mu, \theta) := P_n^l(\cosh \mu) P_n^l(\cos \theta)$ $(l = 0, \ldots, n)$.

In all that follows, we denote by $\boldsymbol{\mathcal{E}}_{n,l}\,(\mu, \theta, \varphi) := \frac{1}{2}\partial[U_{n+1,l}(\mu, \theta) \cos(l\varphi)]$ and $\boldsymbol{\mathcal{F}}_{n,l}\,(\mu, \theta, \varphi) := \frac{1}{2}\partial[U_{n+1,l}(\mu, \theta) \sin(l\varphi)]$. Having in mind the mentioned Laplacian factorization, it is easily seen that $\boldsymbol{\mathcal{E}}_{n,l}$ and $\boldsymbol{\mathcal{F}}_{n,l}$ are monogenic (more specifically, they are a total of $2n+3$ (R)-non-homogeneous polynomial solutions with values in $\mathcal{A}$). These inner solid (prolate) spheroidal monogenics are of the form

$$\begin{aligned} \boldsymbol{\mathcal{E}}_{n,l}\,(\mu, \theta, \varphi) \;=\; & \frac{(n+l+1)}{2} A_{n,l}(\mu, \theta) \cos(l\varphi) \\ & + \frac{1}{4\,(n-l+1)} A_{n,l+1}(\mu, \theta) \left[\cos((l+1)\varphi)\mathbf{i} + \sin((l+1)\varphi)\mathbf{j}\right] \\ & + \frac{1}{4}\,(n+1+l)(n+l)(n-l+2)\, A_{n,l-1}(\mu, \theta) \left[-\cos((l-1)\varphi)\mathbf{i}\right. \\ & \left. + \sin((l-1)\varphi)\mathbf{j}\right], \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \boldsymbol{\mathcal{F}}_{n,l}\,(\mu, \theta, \varphi) \;=\; & \frac{(n+l+1)}{2} A_{n,l}(\mu, \theta) \sin(l\varphi) \\ & + \frac{1}{4\,(n-l+1)} A_{n,l+1}(\mu, \theta) \left[\sin((l+1)\varphi)\mathbf{i} - \cos((l+1)\varphi)\mathbf{j}\right] \\ & - \frac{1}{4}\,(n+1+l)(n+l)(n-l+2)\, A_{n,l-1}(\mu, \theta) \left[\sin((l-1)\varphi)\mathbf{i}\right. \\ & \left. + \cos((l-1)\varphi)\mathbf{j}\right] \end{aligned} \tag{3.2}$$

for $l = 0, \ldots, n+1$ $(n = 0, 1, \ldots)$, with the notation

$$A_{n,l}(\mu, \theta) := \sum_{k=0}^{\left[\frac{n-l}{2}\right]} \frac{(2n+1-4k)\,(n+l-2k+1)_{2k}}{(n-l-2k+1)_{2k+1}} U_{n-2k,l}(\mu, \theta) \tag{3.3}$$

such that

$$A_{n,-1}(\mu,\theta) := \begin{cases} -\frac{1}{n(n+1)^2(n+2)} A_{n,1}(\mu,\theta) & n = 1,2,\dots \\ 0 & n = 0. \end{cases}$$

As usual, for $b \in \mathbb{R}$, $(b)_r = b(b+1)(b+2)\cdots(b+r-1)$ the Pochhammer symbol with $(b)_0 := 1$.

Because of properties of the Ferrer and sine functions, it is easy to see that $A_{n,l}(\mu,\theta) = 0$ for $l > n$ and $\boldsymbol{\mathcal{F}}_{n,0} = 0$ for all n.

For the usual applications we define the previous polynomials in a spheroid which has an unbounded boundary, because $P^l_{n-2k}(\cosh\mu)$ becomes infinite with μ.

In particular, in [21, 23] it is proved that the system

$$\{\boldsymbol{\mathcal{E}}_{n,l},\ \boldsymbol{\mathcal{F}}_{n,l} : l = 0,\dots,n+1;\ n = 0,1,\dots\}$$

forms a complete orthogonal system for the interior of the prolate spheroid (1.2) in the sense of the scalar inner product (2.3). This system can be seen as a refinement and extension of the harmonic polynomial systems exploited by P. Garabedian in [7], and correspondingly it constitutes an extension of the role of the Chebyshev polynomials and Ferrer's associated Legendre functions. The principal point of interest is that the orthogonality of the polynomials does not depend on the shape of the spheroids, but only on the location of the foci of the ellipse generating the spheroid. It is shown a corresponding orthogonality over the surface of these spheroids with respect to a suitable weight function. Properties and applications of these polynomials can be found in [21], [24] and [26].

Remark 3.1. A complete orthogonal system of 3D inner solid spheroidal monogenics for an oblate spheroid have been exploited in [29]. Symbolically, inner solid oblate spheroidal monogenics can be given in a similar way as (3.1)-(3.2) by making the change of "variables" $P^m_{n-2k}(\cosh\mu) \rightarrow i^{n-2k-m} P^m_{n-2k}(i\sinh\mu)$ in the subscript coefficient function (3.3).

Remark 3.2. The corresponding results for the important limiting case, the sphere (of radius $\cosh\mu \simeq \sinh\mu$), can be obtained by a simple transformation.

4. Homogeneous monogenic functions on $\mathbb{R}^3\backslash\{0\}$

Before going further, we take a look at a complete system of the monogenic function space on $\mathbb{R}^3\backslash\{0\}$. In [2], Bock used the Kelvin transformation directly on the Appell system inside the unit ball to construct a complete system on the exterior domain which consists of homogeneous functions. Since it requires full quaternions, this idea is not applicable in the case of reduced quaternions. Thus we shall apply again the harmonic function approach to obtain a complete system of homogeneous monogenic functions on $\mathbb{R}^3\backslash\{0\}$.

Definition 4.1. Given a function u defined on $\mathbb{R}^3$, then the function $K[u]$ defined on $\mathbb{R}^3\backslash\{0\}$ by

$$K[u](x) := \frac{1}{|x|} u\left(\frac{x}{|x|^2}\right)$$

is called the *the Kelvin transformation* of u.

The Kelvin transformation preserves harmonicity, thus we have a complete system of harmonic functions on $\mathbb{R}^3\backslash\{0\}$

$$\left\{\frac{1}{r^{n+1}}P_n(\cos\theta),\ \frac{1}{r^{n+1}}P_n^m(\cos\theta)\cos(m\varphi),\ \frac{1}{r^{n+1}}P_n^m(\cos\theta)\sin(m\varphi)\right\}$$

with $n = 0,1,\ldots;\ m = 1,\ldots,n$. Applying the hypercomplex derivative $\frac{1}{2}\partial$ in spherical coordinates as in [2], we obtain a complete orthogonal system of monogenic functions defined on $\mathbb{R}^3\backslash\{0\}$:

$$\begin{aligned}
X^0_{-(n+2)} &= -\frac{n+1}{2}\frac{P_{n+1}(\cos\theta)}{r^{n+2}} \\
&\quad -\frac{1}{2}\frac{P^1_{n+1}(\cos\theta)}{r^{n+2}}[\cos\varphi\mathbf{i}+\sin\varphi\mathbf{j}] \\
X^m_{-(n+2)} &= -\frac{n-m+1}{2}\frac{P^m_{n+1}(\cos\theta)}{r^{n+2}}\cos(m\varphi) \\
&\quad -\frac{1}{4}\frac{P^{m+1}_{n+1}(\cos\theta)}{r^{n+2}}[\cos((m+1)\varphi)\mathbf{i}+\sin((m+1)\varphi)\mathbf{j}] \\
&\quad +\frac{(n-m+1)(n-m+2)}{4}\frac{P^{m-1}_{n+1}(\cos\theta)}{r^{n+2}}[\cos((m-1)\varphi)\mathbf{i}-\sin((m-1)\varphi)\mathbf{j}] \\
Y^m_{-(n+2)} &= -\frac{n-m+1}{2}\frac{P^m_{n+1}(\cos\theta)}{r^{n+2}}\sin(m\varphi) \\
&\quad -\frac{1}{4}\frac{P^{m+1}_{n+1}(\cos\theta)}{r^{n+2}}[\sin((m+1)\varphi)\mathbf{i}-\cos((m+1)\varphi)\mathbf{j}] \\
&\quad +\frac{(n-m+1)(n-m+2)}{4}\frac{P^{m-1}_{n+1}(\cos\theta)}{r^{n+2}}[\sin((m-1)\varphi)\mathbf{i}+\cos((m-1)\varphi)\mathbf{j}]
\end{aligned}$$

for $n = 0,1,\ldots;\ m = 1,\ldots,n$. These functions are homogeneous with degree of homogenuity $-(n+2)$. The point is that these functions form an orthogonal system with the inner product (2.3) only if Ω is the exterior domain of a ball centering at origin, but this is not the case. Therefore, we will construct a new one and then compare it with the complete system of homogeneous functions in the next section.

5. 3D OUTER SOLID PROLATE SPHEROIDAL MONOGENICS

5.1. **A system of outer solid prolate spheroidal monogenics.**

This subsection introduces an orthogonal system of 3D prolate spheroidal monogenics as solutions to the (R) system for the space exterior of a (prolate) spheroid.

We here borrow from the techniques used in the earlier works [9, 27], and extend those results.

Definition 5.1. Let μ_0 be the value of μ on a fixed prolate spheroidal surface. For the space exterior to the prescribed prolate spheroid $\mu = \mu_0$ we define

$$\widehat{\boldsymbol{\mathcal{E}}_{-1,0}}(\mu,\theta,\varphi) := \frac{-\sinh\mu\cos\theta+\cosh\mu\sin\theta\,(\cos\varphi\mathbf{i}+\sin\varphi\mathbf{j})}{\sinh\mu\,(\sin^2\theta+\sinh^2\mu)}, \tag{5.1}$$

$$\begin{aligned}
\widehat{\boldsymbol{\mathcal{E}}_{0,0}}(\mu,\theta,\varphi) &:= \frac{1}{4}\ln\left(\frac{\cosh\mu+1}{\cosh\mu-1}\right)-\frac{1}{2}\frac{\cosh\mu}{\sin^2\theta+\sinh^2\mu} \\
&\quad +\frac{1}{2}\frac{\sin\theta\cos\theta}{\sinh\mu\,(\sin^2\theta+\sinh^2\mu)}(\cos\varphi\mathbf{i}+\sin\varphi\mathbf{j}),
\end{aligned}\tag{5.2}$$

$$(5.3)\quad \widehat{\boldsymbol{\mathcal{E}}_{n,l}}\,(\mu,\theta,\varphi) \;:=\; \frac{(n+l+1)}{2}\,B_{n,l}(\mu,\theta)\,\cos(l\varphi) + \frac{1}{4\,(n-l+1)}\,B_{n,l+1}(\mu,\theta)\,[\cos((l+1)\varphi)\mathbf{i}+\sin((l+1)\varphi)\mathbf{j}] + \frac{1}{4}\,(n+1+l)(n+l)(n-l+2)\,B_{n,l-1}(\mu,\theta)\,[-\cos((l-1)\varphi)\mathbf{i} + \sin((l-1)\phi)\mathbf{j}]\,,$$

$$(5.4)\quad \widehat{\boldsymbol{\mathcal{F}}_{n,l}}\,(\mu,\theta,\varphi) \;:=\; \frac{(n+l+1)}{2}\,B_{n,l}(\mu,\theta)\sin(l\varphi) + \frac{1}{4\,(n-l+1)}\,B_{n,l+1}(\mu,\theta)\,[\sin((l+1)\phi)\mathbf{i}-\cos((l+1)\varphi)\mathbf{j}] - \frac{1}{4}\,(n+1+l)(n+l)(n-l+2)\,B_{n,l-1}(\mu,\theta)\,[\sin((l-1)\varphi)\mathbf{i} + \cos((l-1)\varphi)\mathbf{j}]\,,$$

(for $l=0,\dots,n\;;\quad n=1,2,\dots;$)

$$(5.5)\quad \widehat{\boldsymbol{\mathcal{E}}_{n,n+1}}\,(\mu,\theta,\varphi) \;:=\; (n+1)B_{n,n+1}(\mu,\theta)\cos((n+1)\varphi) - \frac{\cosh\mu P^{n+2}_{n+2}(\cos\theta)Q^{n+2}_{n+1}(\cosh\mu)}{4(2n+3)(\sin^2\theta+\sinh^2\mu)}\,[\cos((n+2)\phi)\mathbf{i}+\sin((n+2)\varphi)\mathbf{j}] + \frac{(2n+2)(2n+1)}{4}\,B_{n,n}(\mu,\theta)\,[-\cos(n\varphi)\mathbf{i}+\sin(n\phi)\mathbf{j}]\,,$$

$$(5.6)\quad \widehat{\boldsymbol{\mathcal{F}}_{n,n+1}}\,(\mu,\theta,\varphi) \;:=\; (n+1)B_{n,n+1}(\mu,\theta)\sin((n+1)\varphi) - \frac{\cosh\mu P^{n+2}_{n+2}(\cos\theta)Q^{n+2}_{n+1}(\cosh\mu)}{4(2n+3)(\sin^2\theta+\sinh^2\mu)}\,[\sin((n+2)\phi)\mathbf{i}-\cos((n+2)\varphi)\mathbf{j}] - \frac{(2n+2)(2n+1)}{4}\,B_{n,n}(\mu,\theta)\,[\sin(n\varphi)\mathbf{i}+\cos(n\phi)\mathbf{j}]\,,$$

(for $n=0,1,\dots$)

The subscript coefficient function

$$(5.7)\; B_{n,l}(\mu,\theta) \;:=\; \frac{1}{\sin^2\theta+\sinh^2\mu}\,\big[\cosh\mu\,P^l_n(\cos\theta)\,Q^l_{n+1}(\cosh\mu) - \cos\theta\,P^l_{n+1}(\cos\theta)\,Q^l_n(\cosh\mu)\big]\,.$$

where

$$B_{n,-1}(\mu,\theta) := -\frac{1}{n(n+1)^2(n+2)}\,B_{n,1}(\mu,\theta) \qquad \text{for}\;\; n=1,2,\dots$$

We underline that the functions (5.1)–(5.6) is obtained by applying the hypercomplex derivative $\left(\frac{1}{2}\partial\right)$ to harmonic functions $V_{n+1,l}\cos(l\varphi)$ and $V_{n+1,l}\sin(l\varphi)$, where $V_{n+1,l}=Q^l_{n+1}(\cosh\mu)P^l_{n+1}(\cos\theta)$, then they are monogenic functions. Moreover these are not well-defined at the origin since $Q^l_n(\cosh\mu)$ becomes logarithmically infinite when $\mu=0$. The asymptotic behavior of these functions when μ tends to infinity will be discussed later.

It is easily seen that $\widehat{\boldsymbol{\mathcal{F}}_{n,0}} = 0$ for all n as 3D inner spheroidal monogenics. Differently, $\widehat{\boldsymbol{\mathcal{E}}_{n,n+1}}$ and $\widehat{\boldsymbol{\mathcal{F}}_{n,n+1}}$ still have scalar part since $B_{n,n+1}(\mu,\theta) \neq 0$. Indeed, we have

$$B_{n,n+1} = -\frac{\cos\theta P_{n+1}^{n+1}(\cos\theta) Q_n^{n+1}(\cosh\mu)}{\sin^2\theta + \sinh^2\mu}.$$

It is noted that $B_n^l = 0$ for $l \geq n+2$. In other cases, $B_{n,l}(\mu,\theta)$ can be explicitly described using a similar recurrence formula as in [21]

$$B_{n,l}(\mu,\theta) = \frac{2n+1}{n-l+1} P_n^l(\cos\theta) Q_n^l(\cosh\mu) + \frac{(n+l)(n+l-1)}{(n-l+1)(n-l)} B_{n-2,l}(\mu,\theta),$$

with initial values (cf. [37])

$$\begin{aligned} B_{l,l}(\mu,\theta) &= (2l+1) P_l^l(\cos\theta) Q_l^l(\cosh\mu) - 2l\,\frac{\cosh\mu P_l^l(\cos\theta) Q_{l-1}^l(\cosh\mu)}{\sin^2\theta + \sinh^2\mu}, \\ B_{l+1,l}(\mu,\theta) &= \frac{2l+3}{2} P_{l+1}^l(\cos\theta) Q_{l+1}^l(\cosh\mu) - l(2l+1)\,\frac{\cos\theta P_l^l(\cos\theta) Q_{l-1}^l(\cosh\mu)}{\sin^2\theta + \sinh^2\mu} \end{aligned}$$

for $l > 0$ and for $l = 0$

$$\begin{aligned} B_{0,0}(\mu,\theta) &= P_0(\cos\theta) Q_0(\cosh\mu) - \frac{\cosh\mu}{\sin^2\theta + \sinh^2\mu}, \\ B_{1,0}(\mu,\theta) &= \frac{3}{2} P_1(\cos\theta) Q_1(\cosh\mu) - \frac{1}{2}\frac{\cos\theta}{\sin^2\theta + \sinh^2\mu}. \end{aligned}$$

Solving the inductive formula for $B_{n,l}$ leads to

$$\begin{aligned} B_{n,l}(\mu,\theta) = & \sum_{k=0}^{[\frac{n-l}{2}]-1} \frac{(2n+1-4k)\,(n+l-2k+1)_{2k}}{(n-l-2k+1)_{2k+1}}\, P_{n-2k}^l(\cos\theta) Q_{n-2k}^l(\cosh\mu) \\ & + \begin{cases} \dfrac{(2l+1)_{n-l}}{(n-l+1)!} B_{l,l}(\mu,\theta) & \text{if } n-l \text{ even} \\ \dfrac{2(2l+2)_{n-l-1}}{(n-l+1)!} B_{l+1,l}(\mu,\theta) & \text{if } n-l \text{ odd.} \end{cases} \end{aligned} \tag{5.8}$$

These functions $B_{n,l}(\mu,\theta)$ play an important role in discovering properties of our monogenic function system. Step by step, we will discuss more about their characteristics.

5.2. **Asymptotic behavior.** Our aim is constructing an orthogonal complete system of the L_2-monogenic function space in the exterior domain of a prolate spheroid. Therefore at least these functions must tend to zero at infinity. Theoretically, a basis of monogenic functions can be constructed by using *Kelvin transformation* as presented in [2]. The obtained functions are *homogeneous of degree* $-k$ (in Cartesian coordinates) with $k \geq 2$, which defines their asymptotic behaviors. Unfortunately, such a basis is only orthogonal in spherical cases. In the next section, we will prove the orthogonality. Now, it would be nice to know how our functions behave at infinity. In order to find the answer, we look back to Legendre functions of the second kind. For $|z| > 1$

$$Q_n^l(z) = (-1)^l e^{i\pi l} \frac{\sqrt{\pi}\Gamma(n+l+1)}{2^{n+1}\Gamma(n+\frac{3}{2})} \frac{(z^2-1)^{\frac{l}{2}}}{z^{n+l+1}} F\left(\frac{2+n+l}{2}, \frac{1+n+l}{2}, \frac{2n+3}{2}; \frac{1}{z^2}\right)$$

where $\Gamma(t)$ and $F(\alpha, \beta, \gamma; t)$ are the gamma and hypergeometric functions, respectively (see also [30]). When z tends to infinity

$$Q_n^l(z) \simeq \frac{(n+l)!}{(2n+1)!!} \frac{1}{z^{n+1}}.$$

Now let $z = \cosh\mu \simeq \sinh\mu \simeq r$ when μ is large enough (with $r = \sqrt{x_0^2 + x_1^2 + x_2^2}$), it leads to

$$\begin{aligned} B_{n,l}(\mu, \theta) &\simeq \frac{1}{r^2}\left[r P_n^l(\cos\theta) \frac{(n+l+1)!}{(2n+3)!!} \frac{1}{r^{n+2}} - \cos\theta P_{n+1}^l(\cos\theta) \frac{(n+l)!}{(2n+1)!!} \frac{1}{r^{n+1}} \right] \\ &= -\frac{(n-l+2)(n+l)!}{(2n+3)!!} \frac{P_{n+2}^l(\cos\theta)}{r^{n+3}}. \end{aligned}$$

As a result, we obtain the asymptotic behaviors of $\widehat{\boldsymbol{\mathcal{E}}_{n,l}}$ and $\widehat{\boldsymbol{\mathcal{F}}_{n,l}}$ for $l = 0, \ldots, n+1$; $n = 0, 1, \ldots$

$$\begin{aligned} \widehat{\boldsymbol{\mathcal{E}}_{n,l}} &\simeq \frac{(n+l+1)!}{(2n+3)!!} X_{-(n+3)}^l, \\ \widehat{\boldsymbol{\mathcal{F}}_{n,l}} &\simeq \frac{(n+l+1)!}{(2n+3)!!} Y_{-(n+3)}^l. \end{aligned}$$

One can see that at infinity $\widehat{\boldsymbol{\mathcal{E}}_{n,l}}$ and $\widehat{\boldsymbol{\mathcal{F}}_{n,l}}$ behave like homogeneous monogenic functions of degree $-(n+3)$. A special case corresponding to $\widehat{\boldsymbol{\mathcal{E}}_{-1,0}}$ when $\mu \to \infty$

$$\widehat{\boldsymbol{\mathcal{E}}_{-1,0}} \simeq -\frac{1}{r^2}\left[\cos\theta - \sin\theta\left(\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j}\right)\right] = -\frac{\overline{x}}{|x|^3}.$$

It means that $\widehat{\boldsymbol{\mathcal{E}}_{-1,0}}$ has the asymptotic property similar to *the Cauchy kernel* in a neighborhood of infinity. This is very important not only to ensure that those functions are well defined on the exterior domain of a prolate spheroid, but also it gives us an evidence of the completeness of such a system.

5.3. **Orthogonality.** In order to make a concise proof about the orthogonality of the system, we formulate at first some supplementary results.

Proposition 5.2. Let $\widehat{\boldsymbol{\mathcal{E}}_{n,m}}$, $\widehat{\boldsymbol{\mathcal{F}}_{k,l}}$ be functions as described in (5.1)–(5.6). Each following pair of functions are orthogonal with respect to the inner product (2.3) whenever $l_1 \neq l_2$:

- $\{\widehat{\boldsymbol{\mathcal{E}}_{n_1,l_1}}, \widehat{\boldsymbol{\mathcal{E}}_{n_2,l_2}}\}$
- $\{\widehat{\boldsymbol{\mathcal{F}}_{n_1,l_1}}, \widehat{\boldsymbol{\mathcal{F}}_{n_2,l_2}}\}$
- $\{\widehat{\boldsymbol{\mathcal{E}}_{n_1,l_1}}, \widehat{\boldsymbol{\mathcal{F}}_{n_2,l_2}}\}$

We skip the proof here. In fact, the 3D outer monogenic functions (5.1)–(5.6) share the same structure with the 3D inner monogenic functions (3.1)-(3.2) which have been studied in several articles by J. Morais, among other mathematicians.

Therefore, the orthogonality in cases of different orders $l_1 \neq l_2$ can be done similarly. We refer readers to [21, 23, 29] for a detail proof.

The difference between the 3D inner and outer monogenics comes from the supplementary functions $A_{n,l}(\mu,\theta)$ in (3.3) and $B_{n,l}(\mu,\theta)$ in (5.7). As we can see, $A_{n,l}(\mu,\theta)$ is decomposed into summands of the form

$$P^l_{n-2k}(\cosh\mu)P^l_{n-2k}(\cos\theta).$$

Consequently, the remaining proof for cases of different degrees $n_1 \neq n_2$ is done due to the orthogonality of associated Legendre polynomials of the first kind

$$\int_0^\pi P^l_n(\cos\theta)\, P^l_s(\cos\theta)\, \sin\theta\, d\theta \;=\; 0,$$

$$\int_0^\pi P^l_{n+1}(\cos\theta)\, \cos\theta\, P^l_s(\cos\theta) \sin\theta\, d\theta \;=\; 0$$

where $s < n$. However, $B_{n,l}(\mu,\theta)$ includes not only summands

$$Q^l_{n-2k}(\cosh\mu)P^l_{n-2k}(\cos\theta)$$

but also one extra–term

$$\frac{\cosh\mu P^l_l(\cos\theta)Q^l_{l-1}(\cosh\mu)}{\sin^2\theta + \sinh^2\mu} \qquad \text{or} \qquad \frac{\cos\theta P^l_l(\cos\theta)Q^l_{l-1}(\cosh\mu)}{\sin^2\theta + \sinh^2\mu}.$$

For this reason, it is necessary to improve our techniques to overcome such a difference. We express an orthogonal property for the ansatz function $B_{n,l}(\mu,\theta)$ by the underline proposition:

Proposition 5.3. Let $B_{n,l}(\mu,\theta)$ be functions as in the Definition 5.1, then with $l = 0, 1, \dots$ the following equalities hold for each pair of $(n,\, k) :\; k, n \in \{l,\, l+1\}$

$$\int_0^\pi B_{n,l}(\mu,\theta)\, P^l_k(\cos\theta)\, \sin\theta\, d\theta \;=\; 0.$$

Proof. In fact, there are 4 equalities corresponding to 4 choices of (n,k). We put a parameter χ taking values in the set $\{(0,0),\, (0,1),\, (1,0),\, (1,1)\}$ to express choices of (n,k). For example, $\chi = (1,0)$ corresponds to $n = l+1$ and $k = l$. We denote left hand sides of the equalities by I^l_χ. Note that we always have

$$\mathrm{I}^l_{(0,1)} = \mathrm{I}^l_{(1,0)} = 0.$$

Indeed, let us consider for example

$$\begin{aligned}
\mathrm{I}^l_{(1,0)} &= \int_0^\pi B_{l+1,l}(\mu,\theta)\, P^l_l(\cos\theta)\, \sin\theta\, d\theta \\
&= \int_0^\pi \left\{ \frac{2l+3}{2} P^l_{l+1}(\cos\theta)Q^l_{l+1}(\cosh\mu) - l\, \frac{P^l_{l+1}(\cos\theta)Q^l_{l-1}(\cosh\mu)}{\sin^2\theta + \sinh^2\mu} \right\} \\
&\qquad\qquad \times P^l_l(\cos\theta)\, \sin\theta\, d\theta.
\end{aligned}$$

The first term has zero–integral because of the orthogonality of associated Legendre polynomials. The second term has also vanishing integral because this is an odd function with respect to variable $t := \cos\theta$.

Now we consider the two remaining integrals $\mathrm{I}^l_{(0,0)}$ and $\mathrm{I}^l_{(1,1)}$. Basically, there is no difference between them. Thus now we prove for $\mathrm{I}^l_{(0,0)}$. The other can be derived analogously.

First of all, let us calculate $\mathrm{I}^l_{(0,0)}$ for some initial values $l = 0, 1$

$$\begin{aligned}\mathrm{I}^0_{(0,0)} &= \int_0^\pi B_{0,0}(\mu,\theta)\, P_0(\cos\theta)\, \sin\theta\, d\theta \\ &= \int_0^\pi \left\{ Q_0(\cosh\mu) - \frac{\cosh\mu}{\sin^2\theta + \sinh^2\mu} \right\} \sin\theta\, d\theta \\ &= 2Q_0(\cosh\mu) \,-\, \ln\left(\frac{\cosh\mu + 1}{\cosh\mu - 1}\right) \,=\, 0,\end{aligned}$$

$$\begin{aligned}\mathrm{I}^1_{(0,0)} &= \int_0^\pi B_{1,1}(\mu,\theta)\, P^1_1(\cos\theta)\, \sin\theta\, d\theta \\ &= \int_0^\pi \left\{ 3P^1_1(\cos\theta)Q^1_1(\cosh\mu) - 2\,\frac{\cosh\mu P^1_1(\cos\theta)Q^1_0(\cosh\mu)}{\sin^2\theta + \sinh^2\mu} \right\} P^1_1(\cos\theta)\, \sin\theta\, d\theta \\ &= 4Q^1_1(\cosh\mu) - 2\cosh\mu Q^1_0(\cosh\mu) \int_0^\pi \frac{\sin^2\theta}{\sin^2\theta + \sinh^2\mu} \sin\theta\, d\theta \\ &= 0.\end{aligned}$$

For arbitrary $l > 1$, one has

$$\begin{aligned}\mathrm{I}^l_{(0,0)} &= \int_0^\pi B_{l,l}(\mu,\theta)\, P^l_l(\cos\theta)\, \sin\theta\, d\theta \\ &= \int_0^\pi \left\{ (2l+1)P^l_l(\cos\theta)Q^l_l(\cosh\mu) - 2l\,\frac{\cosh\mu P^l_l(\cos\theta)Q^l_{l-1}(\cosh\mu)}{\sin^2\theta + \sinh^2\mu} \right\} \\ &\qquad\qquad \times P^l_l(\cos\theta)\, \sin\theta\, d\theta \\ &= 2(2l)!Q^l_l(\cosh\mu) - 2l\cosh\mu Q^l_{l-1}(\cosh\mu) \int_0^\pi \frac{[P^l_l(\cos\theta)]^2}{\sin^2\theta + \sinh^2\mu} \sin\theta\, d\theta.\end{aligned}$$

Now we rewrite the integral part in the above formula

$$\begin{aligned}\int_0^\pi \frac{[P^l_l(\cos\theta)]^2}{\sin^2\theta + \sinh^2\mu} \sin\theta\, d\theta &= (2l-1)^2 \int_0^\pi \frac{\sin^2\theta [P^{l-1}_{l-1}(\cos\theta)]^2}{\sin^2\theta + \sinh^2\mu} \sin\theta\, d\theta \\ &= 2(2l-1)! - (2l-1)^2\sinh^2\mu \int_0^\pi \frac{[P^{l-1}_{l-1}(\cos\theta)]^2}{\sin^2\theta + \sinh^2\mu} \sin\theta\, d\theta.\end{aligned}$$

Moreover from properties of associated Legendre functions of the second kind (see [37]), one gets

$$\sinh\mu\, Q^l_{l-1}(\cosh\mu) = 2(l-1)Q^{l-1}_{l-2}(\cosh\mu).$$

Subtitute all these stuffs into the calculation of $\mathrm{I}^l_{(0,0)}$, we find that

$$\begin{aligned}\mathrm{I}^l_{(0,0)} &= -2l(2l-1)^2 \sinh\mu\, \mathrm{I}^{l-1}_{(0,0)} \\ &\quad + 2(2l)!\, \underbrace{\left[Q^l_l(\cosh\mu) - \cosh\mu Q^l_{l-1}(\cosh\mu) + (2l-1)\sinh\mu\, Q^{l-1}_{l-1}(\cosh\mu)\right]}_{=0} \\ &= -2l(2l-1)^2 \sinh\mu \mathrm{I}^{l-1}_{(0,0)}\end{aligned}$$

This is an inductive formula and associated with initial values $\mathrm{I}^0_{(0,0)} = \mathrm{I}^1_{(0,0)} = 0$, it yields that $\mathrm{I}^l_{(0,0)} = 0$ for every $l = 0, 1, \ldots$ It completes the proof. $\square$

To end this section, we conclude by a theorem of the orthogonality of the function system (5.1)–(5.6)

Theorem 5.4. *The constructed functions (5.1)–(5.6) form an orthogonal system with respect to the inner product* (2.3)*.*

Proof. As we discussed in the proposition 5.2, two functions in (5.1)–(5.6) are orthogonal if they have different orders $l_1 \neq l_2$. It is now to prove the orthogonality in cases of the same order l. Look back to the form of functions (5.1)–(5.6), we see that $\widehat{\boldsymbol{\mathcal{E}}_{n,l}}$ and $\widehat{\boldsymbol{\mathcal{F}}_{k,l}}$ are always orthogonal because of the orthogonality of following pairs

$$\{\sin(l\varphi),\, \cos(l\varphi)\}, \quad \{\sin[(l+1)\varphi],\, \cos[(l-1)\varphi]\}, \quad \{\sin[(l-1)\varphi],\, \cos[(l+1)\varphi]\}$$

when φ runs from 0 to 2π. It is enough to prove such a property inside each subset of $\{\widehat{\boldsymbol{\mathcal{E}}_{n,l}}\}$ and $\{\widehat{\boldsymbol{\mathcal{F}}_{k,l}}\}$ functions. Remark that in the special cases of $\widehat{\boldsymbol{\mathcal{E}}_{-1,0}}$, $\widehat{\boldsymbol{\mathcal{E}}_{n,n+1}}$ and $\widehat{\boldsymbol{\mathcal{F}}_{n,n+1}}$ the following arguments still hold. Thus we just prove for the general form of $\widehat{\boldsymbol{\mathcal{E}}_{n,l}}$ and $\widehat{\boldsymbol{\mathcal{F}}_{k,l}}$. Now, if we calculate the inner product of these functions and cancel vanishing–terms, the remains are integrals of the forms

$$\int_0^{\pi} B_{n_1,m}(\mu,\theta)\, B_{n_2,m}(\mu,\theta) \sin\theta(\sin^2\theta + \sinh^2\mu)d\theta$$

where $m = l-1,\, l,\, l+1$. Suppose that $n_1 > n_2$. Present $B_{n_1,m}(\mu,\theta)$ as (5.7) and $B_{n_2,m}(\mu,\theta)$ as (5.8) and omit vanishing–integrals, what left are

$$\int_0^{\pi} B_{n_1,m}(\mu,\theta)\, P_l^l(\cos\theta) \sin\theta d\theta$$

if $n_2 - m$ even, or

$$\int_0^{\pi} B_{n_1,m}(\mu,\theta)\, P_{l+1}^l(\cos\theta) \sin\theta d\theta$$

if $n_2 - m$ odd. Using again (5.8) for $B_{n_1,m}(\mu,\theta)$, it leads to integrals of the form as stated in the proposition 5.3 and so remaining integrals vanish. Hence the theorem follows. $\square$

5.4. **Completeness.** Now we claim the main theorem of this paper.

Theorem 5.5. *The functions (5.1)–(5.6) form an orthogonal complete system in the space* $\mathcal{M}(\Omega^-, \mathcal{A})$*.*

Proof. We have already proved the orthogonality. To prove the completeness, we only need to prove that every function in $\{X^0_{-(n+2)}, X^m_{-(n+2)}, Y^m_{-(n+2)} : n = 0, 1, \ldots ; m = 1, \ldots, n\}$ can be expressed by our function system. Indeed, let us consider $X^m_{-(n+2)}$ as an example. We have

$$X^m_{-(n+2)} = \frac{1}{2}\partial \left[\frac{1}{r^{n+1}} P_n^m(\cos\theta) \cos(m\varphi) \right].$$

The term inside the square brackets $[\cdot]$ is a harmonic function defined on Ω^-. Thus it can be expressed in terms of outer solid spheroidal harmonic functions

$$\sum_{n=0}^{\infty}\Big(a_n^0 Q_n(\cosh\mu)P_n(\cos\theta)+\sum_{m=1}^{n}Q_n^m(\cosh\mu)P_n^m(\cos\theta)$$
$$\times\left[a_n^m\cos(m\varphi)+b_n^m\sin(m\varphi)\right]\Big).$$

This series expansion converges in $L_2(\Omega^-)$. Differentiate summand by summand, one finally gets

$$X^m_{-(n+2)}=a_0^0\widehat{\boldsymbol{\mathcal{E}}_{-1,0}}+b_1^1\widehat{\boldsymbol{\mathcal{F}}_{0,1}}+\sum_{n=0}^{\infty}\sum_{m=0}^{n+1}a^m_{n+1}\widehat{\boldsymbol{\mathcal{E}}_{n,m}}+\sum_{n=1}^{\infty}\sum_{m=1}^{n+1}b^m_{n+1}\widehat{\boldsymbol{\mathcal{F}}_{n,m}}.$$

Apply the same arguments to other homogeneous monogenic functions, the theorem then follows. □

5.5. **General properties.** We begin this subsection by summarizing some basic properties of the basis functions.

Theorem 5.6. *Let μ_0 be the value of μ on a fixed prolate spheroidal surface. The outer solid prolate spheroidal monogenics (5.1)–(5.6) satisfy the following properties:*

(1) $\widehat{\boldsymbol{\mathcal{E}}_{n,l}}$ *and* $\widehat{\boldsymbol{\mathcal{F}}_{n,m}}$ *are the zero functions for* $l,m\geq n+2$*;*
(2) $\widehat{\boldsymbol{\mathcal{E}}_{n,l}}$ *and* $\widehat{\boldsymbol{\mathcal{F}}_{n,m}}$ *are* 2π*-periodic with respect to the variable* ϕ*;*
(3) *For each* $n\in\mathbb{N}$*, the inhomogeneous harmonic functions* $\mathrm{Sc}(\widehat{\boldsymbol{\mathcal{E}}_{n,l}})$ $(l=0,\dots,n)$ *and* $\mathrm{Sc}(\widehat{\boldsymbol{\mathcal{F}}_{n,m}})$ $(m=1,\dots,n)$ *form an orthogonal system over the space exterior to* μ_0 *in the sense of the product (2.3);*
(4) *For each* $n\in\mathbb{N}$*, each of the two sets*

$$\Big\{\mathrm{Sc}(\widehat{\boldsymbol{\mathcal{E}}_{n,l}}),\,[\widehat{\boldsymbol{\mathcal{E}}_{n,l}}]_1,\,[\widehat{\boldsymbol{\mathcal{E}}_{n,l}}]_2\,:\,l=0,\dots,n+1\Big\},$$

and

$$\Big\{\mathrm{Sc}(\widehat{\boldsymbol{\mathcal{F}}_{n,m}}),\,[\widehat{\boldsymbol{\mathcal{F}}_{n,m}}]_1,\,[\widehat{\boldsymbol{\mathcal{F}}_{n,m}}]_2\,:\,m=1,\dots,n+1\Big\}$$

forms an orthogonal system over the space exterior to μ_0 *in the sense of the scalar product (2.3).*

Proof. Statements 1. and 2. follow from the properties of the Ferrer's functions and Chebyshev polynomials. The proofs of Statements 3. and 4. are a consequence of Theorem 5.4, and having in mind that $\mathrm{Sc}(\widehat{\boldsymbol{\mathcal{E}}_{n,n+1}})=\mathrm{Sc}(\widehat{\boldsymbol{\mathcal{F}}_{n,n+1}})=0$. □

In view of the practical applicability of the aforementioned spheroidal monogenics, next we illustrate explicit recurrence rules between them that are plain to be integrated in a computational framework.

Proposition 5.7. For each $n \in \mathbb{N}$, the outer solid prolate spheroidal monogenics (5.1)-(5.6) satisfy the recurrence formulae:

$$\begin{aligned}\widehat{\boldsymbol{\mathcal{E}}_{n,l}} &= -\frac{(n+l+1)(n-l+2)}{2}\left(\widehat{\boldsymbol{\mathcal{E}}_{n,l-1}}\mathbf{i}-\widehat{\boldsymbol{\mathcal{F}}_{n,l-1}}\mathbf{j}\right)\\ &\qquad +\frac{(1+\delta_{0,l})}{2\,(n+l+2)(n-l+1)}\left(\widehat{\boldsymbol{\mathcal{E}}_{n,l+1}}\mathbf{i}+\widehat{\boldsymbol{\mathcal{F}}_{n,l+1}}\mathbf{j}\right);\end{aligned}$$

and

$$\begin{aligned}\widehat{\boldsymbol{\mathcal{F}}_{n,m}} &= -\frac{(n+m+1)(n-m+2)}{2}\left(\widehat{\boldsymbol{\mathcal{F}}_{n,m-1}}\mathbf{i}+\widehat{\boldsymbol{\mathcal{E}}_{n,m-1}}\mathbf{j}\right)\\ &\qquad +\frac{1}{2\,(n+m+2)(n-m+1)}\left(\widehat{\boldsymbol{\mathcal{F}}_{n,m+1}}\mathbf{i}-\widehat{\boldsymbol{\mathcal{E}}_{n,m+1}}\mathbf{j}\right)\end{aligned}$$

for $l = 0, \dots, n+1$ and $m = 1, \dots, n+1$, with starting value (5.2). For a more unified formulation we remind the reader that $\widehat{\boldsymbol{\mathcal{E}}_{n,m}} = \widehat{\boldsymbol{\mathcal{F}}_{n,m}} \equiv 0$ for $m \geq n+2$, and $\delta_{l,0}$ is the Kronecker symbol.

Proof. For simplicity we just present the calculations for $\widehat{\boldsymbol{\mathcal{E}}_{n,m}}$ $(m > 1)$. By direct inspection of previous expressions one has

$$\begin{aligned}&-\frac{(n+m+1)(n-m+2)}{2}\left(\widehat{\boldsymbol{\mathcal{E}}_{n,m-1}}\mathbf{i}-\widehat{\boldsymbol{\mathcal{F}}_{n,m-1}}\mathbf{j}\right)\\ &\qquad +\frac{1}{2\,(n+m+2)(n-m+1)}\left(\widehat{\boldsymbol{\mathcal{E}}_{n,m+1}}\mathbf{i}+\widehat{\boldsymbol{\mathcal{F}}_{n,m+1}}\mathbf{j}\right)\\ =\ & \frac{(n+m+1)}{2}\,B_{n,m}(\mu,\theta)\,\cos(m\phi)\\ +\ & \frac{1}{4\,(n-m+1)}\,B_{n,m+1}(\mu,\theta)\,\cos((m+1)\phi)\mathbf{i}\\ -\ & \frac{1}{4}\,(n+1+m)(n+m)(n-m+2)\,B_{n,m-1}(\mu,\theta)\,\cos((m-1)\phi)\mathbf{i}\\ &\quad + \frac{1}{4}\,(n+1+m)(n+m)(n-m+2)\,B_{n,m-1}(\mu,\theta)\sin((m-1)\phi)\mathbf{j}\\ &\quad + \frac{1}{4\,(n-m+1)}\,B_{n,m+1}(\mu,\theta)\,\sin((m+1)\phi)\mathbf{j}\\ &\quad = \widehat{\boldsymbol{\mathcal{E}}_{n,m}}, \quad m = 1, \dots, n+1.\end{aligned}$$

The proofs for $\widehat{\boldsymbol{\mathcal{E}}_{n,0}}$ and $\widehat{\boldsymbol{\mathcal{F}}_{n,m}}$ $(m > 1)$ follow the same principle and are therefore straightforward. □

As a direct consequence, we obtain the following recurrence relation.

Corollary 5.8. For each $n \in \mathbb{N}$, the outer solid prolate spheroidal monogenics satisfy the two-term type recurrence formula:

$$\begin{aligned}&(n+2)(n+1)\,\widehat{\boldsymbol{\mathcal{E}}_{n,0}}-\widehat{\boldsymbol{\mathcal{E}}_{n,1}}\mathbf{i}-\widehat{\boldsymbol{\mathcal{F}}_{n,1}}\mathbf{j} \;=\; 0,\\ &(n+m+1)(n-m+2)\left(\widehat{\boldsymbol{\mathcal{E}}_{n,m-1}}-\widehat{\boldsymbol{\mathcal{F}}_{n,m-1}}\mathbf{k}\right)-\widehat{\boldsymbol{\mathcal{E}}_{n,m}}\mathbf{i}-\widehat{\boldsymbol{\mathcal{F}}_{n,m}}\mathbf{j} \;=\; 0\end{aligned}$$

for $m = 1, \dots, n+1$, with the starting value (5.2).

6. Perspectives and concluding remarks

In the presented study we constructed an explicit complete orthogonal system of (R)-solutions for the space exterior of a 3D prolate spheroid. The system is composed by piecewise monogenic functions, and is capable of describing exactly the general information expressed by the commonly used Ferrer's associated Legendre functions of the first and second kinds, which seems to be suitable for the treatment of monogenic functions by the use of power series expansions. Recurrence formulae for fast computer implementations were also given. We honestly expect that the rising popularity of these generalized spheroidal functions is still on the rise and will likely see even more growth in the future, due to their promising applications in many fields.

Recent studies have shown that our approach is connected, in a systematic fashion, with the problem of scattering for both Dirichlet and Neumann boundary conditions in the case in which the boundary is the prolate spheroid μ_0. We are convinced that such problems can be handled with Fourier techniques by expansion into the above spheroidal monogenics. With the help of the constructed system, we also hope to contribute to questions related to an (orthogonal) type Laurent series expansion and its interplay with the classical Cauchy's integral formula, and a number of underlying applications. Further investigations on this topic are now under investigation and will be reported in a forthcoming paper.

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